Lecture 11

Convex games exercise

Exercise 1

Consider the following 2-player game, Γ:

$$J^i(x^1, x^2) = \frac{1}{2}(x^1)^2 + \frac{1}{2}(x^2)^2 + c\,x^1x^2 + b^i x^i,$$

where $c, b^i \in \mathbb{R}$ are constants and $x^i \in \mathbb{R}$, $i = 1, 2$.

a) Show that the game is convex.

b) Compute the game pseudo-gradient.

c) Under which conditions on c, the pseudo-gradient is monotone?

d) What is the necessary & sufficient for $(x^1, x^2) \in \mathbb{R}^2$ to be a Nash equilibrium of Γ? Under which conditions on c & b^i, $i = 1, 2$, it exists?

e) Argue that if $x^i \in [0, k] \subseteq \mathbb{R}$, $i = 1, 2$. the game always has a Nash equilibrium

Solutions :

a) Notice that $J^i(x^i, x^{-i})$ is convex in x^i for fixed x^{-i}; since

$$\frac{\partial J^i}{\partial x^i}(x^i, x^{-i}) = x^i + c x^{-i} + b^i$$

$$\frac{\partial^2 J^i}{(\partial x^i)^2}(x^i, x^{-i}) = 1 > 0 .$$

Namely, each players' cost is (strongly) convex quadratic in her decision variable.

Furthermore, the action spaces are $K^i = \mathbb{R}$, which is convex

$\Rightarrow$ game is convex.

b) $$F_\Gamma(x^1, x^2) = \begin{bmatrix} \frac{\partial J^1}{\partial x^1}(x^1, x^2) \\ \frac{\partial J^2}{\partial x^2}(x^1, x^2) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix}}_{M \in \mathbb{R}^{2\times 2}} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \underbrace{\begin{bmatrix} b^1 \\ b^2 \end{bmatrix}}_{m \in \mathbb{R}^2}$$

c) $F_\Gamma(x^1, x^2)$ is affine : $F_\Gamma(x) = Mx + b$

It is monotone if M is positive semi-definite. Since M is symmetric, this is equivalent to eigenvalues of M being non-negative.

The eigenvalues of M are given by solutions to the characteristic equation

$$(\lambda - 1)^2 - c^2 = 0$$

$$\lambda^2 - 2\lambda - c^2 + 1 = 0$$

$$\Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4 + 4(c^2 - 1)}}{2} = 1 \pm \sqrt{c^2}$$

For both to be non-negative, we need,

$$|c| < 1 .$$

(d)

$$J^1(x^1, x^2) \leq J^1(\tilde{x}^1, x^2), \ \forall \tilde{x}^1 \in \mathbb{R}$$

$$J^2(x^1, x^2) \leq J^2(x^1, \tilde{x}^2), \ \forall \tilde{x}^2 \in \mathbb{R}$$

from player-wise convexity & $x^i \in \mathbb{R}, i=1,2$, Nash equilibrium if and only if:

$$\nabla_{x^1} J^1(x^1, x^2) = 0$$

$$\nabla_{x^2} J^2(x^1, x^2) = 0$$

$$\Longleftrightarrow \underbrace{\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix}}_{M} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = -\underbrace{\begin{bmatrix} b^1 \\ b^2 \end{bmatrix}}_{m}, \ M \in \mathbb{R}^{2\times 2}$$

we need the above set of linear equations to have a solution. $\Longleftrightarrow$ $m \in \text{Range}(M)$

Note that a sufficient condition for $m \in \text{Range}(M)$ is that M is invertible, namely, no eigenvalues at 0 $\Longleftrightarrow$ $|c| \neq 1$.